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Toggling Independent Sets of a Path Graph

Michael Joseph and Tom Roby

Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009, USA

Abstract. This extended abstract summarizes the results in a recent paper by the authors about the orbit structure and homomesy (constant averages over orbits) properties of certain actions of toggle groups on the collection of independent sets of a path graph. In particular we prove that for the action of a "Coxeter element" of vertex toggles, the difference of indicator functions of symmetrically-located vertices is 0-mesic. Then we use our analysis to show facts about orbit sizes that are easy to conjecture but nontrivial to prove.

Besides its intrinsic interest, this particular combinatorial dynamical system is valuable in providing an interesting example of (a) homomesy in a context where large orbit sizes make a cyclic sieving phenomenon unlikely to exist, (b) the use of Coxeter theory to greatly generalize the set of actions for which our results hold, and (c) the value of Striker's notion of generalized toggle groups.

Keywords: Coxeter groups, homomesy, independent sets, promotion, toggle groups

1 Introduction

This paper explores the orbit structure and homomesy properties of certain actions of toggle groups on the collection of independent sets of a path graph. In particular we prove that for the action of a "Coxeter element" of vertex toggles, the difference of indicator functions of symmetrically-located vertices is 0-mesic (Theorem 2.8). We then use our analysis to show facts about orbit sizes that are easy to conjecture but nontrivial to prove. Refer to the full paper [8] for more detail and for the omitted proofs.

Besides its intrinsic interest, this particular combinatorial dynamical system is valuable in several respects. First, it provides an interesting example of homomesy in a context where unwieldy orbit sizes make a cyclic sieving phenomenon (CSP), in the sense of Reiner, Stanton, and White [12], unlikely to exist. Many combinatorial dynamical systems that have a CSP also have natural homomesic statistics and vice versa, though there appears to be no direct connection between the two.

Second, it displays the usefulness of Striker's notion of generalized toggle groups [17] to settings beyond that of posets. Although there is an equivariant bijection (Proposition 4.6) between the action we study on independent sets and the action of promotion on certain posets, the former setting makes it easier to obtain our results.

Third, by taking a Coxeter theoretic approach, we are able to greatly generalize the set of actions for which our results hold, from the specific action of toggling at each vertex left to right to toggling once per vertex in an arbitrary order.

We now describe the setting and background necessary to understand the problem.

Definition 1.1. Let $\mathcal{P}_n$ denote the **path graph** with n vertices, whose vertex set is $[n] := \{1, 2, \ldots, n\}$ and whose edge set is $\{(i, i+1) : i \in [n-1]\}$. An **independent set** of a graph is a subset of the vertices, no two adjacent. Let $\mathcal{I}_n$ denote the set of independent sets of $\mathcal{P}_n$.

Example 1.2. The set of vertices $\{1, 4, 6\}$ is an independent set of $\mathcal{P}_7$, but $\{1, 4, 5, 6\}$ is not. These are represented (respectively) as

and

Although we sometimes write independent sets as subsets of $[n] := \{1, 2, \ldots, n\}$ as above, it may not be obvious in that notation what the value of n is. Another notation that is often more convenient for an independent set is its **binary representation**, i.e., the characteristic vector of S. For example 0010010 represents the independent set $\{3, 6\}$ of $\mathcal{P}_7$. Thus $\mathcal{I}_n$ is the set of length n binary strings that do not contain the subsequence 11 (which would indicate the inclusion of two adjacent vertices). It is well-known and easy to see that the cardinality of $\mathcal{I}_n$ is a Fibonacci number.

In Section 2, we define the *toggle group* $\mathcal{T}_n$ on $\mathcal{I}_n$, which is generated by involutions. Cameron and Fon-der-Flaass introduced the toggle group for order ideals of a poset [4]. More recently, Striker has extensively studied toggle groups in more generalized settings [17]. More specifically, given a set E and a fixed set of "allowed" subsets $\mathcal{L} \subseteq 2^E$, each element $e \in E$ has an associated toggle map which removes or inserts e into any given set in $\mathcal{L}$ provided the resulting set is still in $\mathcal{L}$, and otherwise does nothing. For us, the ground set is $[n]$ and the set of allowed subsets of $[n]$ is $\mathcal{I}_n$.

Our main theorem is Theorem 2.8, an example of the *homomesy* (Greek for "same middle") phenomenon, which was introduced by Propp and the second author in [11], and defined as follows.

Definition 1.3. Suppose we have a set S, an invertible map $w : S \to S$ such that every w-orbit is finite, and a function ("statistic") $f : S \to \mathbb{K}$, where $\mathbb{K}$ is a field of characteristic 0. Then we say the triple (S, w, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$, $\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c$. In this case, we say that the function f is **homomesic** with average c, or ***c*-mesic**, under the action of w on S.

Some early isolated examples of homomesy exist in the literature, notably in the conjecture of Panyushev [9], which was proved by Armstrong, Stump, and Thomas in [1], but serious investigation of homomesy is quite recent. Examples now include cyclic actions on integer partitions, Suter's action on Young diagrams, rowmotion (and promotion) of order ideals and antichains in nice posets, Lyness 5-cycles (which have strong connections to cluster algebra theory), toggling noncrossing partitions, toggling in order polytopes, and many others [5, 6, 7, 11, 13, 18]. In particular, the examples of homomesy in [6, 18] are for maps defined as products of toggles.

In most cases, the systems displaying homomesy had actions whose order was relatively small (sometimes unexpectedly so, as in the first results for the rowmotion map on the product of two chains [3] relative to naive expectations as well as orbit sizes that are relatively tame). These are also the situations in which cyclic sieving is more common. In the situation of this paper, however, the orbit sizes are unpredictable and lead to the action having large order, making an interesting CSP seem rather unlikely. We will eventually discuss how to determine the sizes of the orbits under our maps, but they do not divide a number that is easy to describe without stating all the orbit sizes. Another example of homomesy in a similar situation, involving toggling of noncrossing partitions appears in [6].

To prove Theorem 2.8, we associate an *orbit board* to each orbit, and partition the 1s in the orbit board into *snakes* which begin in the left column and end in the right column. We will prove that each snake determines the entire orbit, and show how a composition naturally associated to snake cyclically rotates along an orbit of the action. Here we are following the lead of Shahrzad Haddadan, who used a similar technique prove homomesy for the action of "winching" on k-element subsets of $[n]$ [7]. Besides proving homomesy, our snake representations lead to many other consequences of orbits, such as the number of orbits, and their sizes (Section 3).

In Section 4, we will explain how our results can be restated for the rowmotion operator on order ideals of zigzag posets $J(\mathcal{Z}_n)$, via an equivariant bijection ("cryptomorphism") $\eta : \mathcal{I}_n \to J(\mathcal{Z}_n)$.

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2 Toggle Maps on Independent Sets

In this section we state and prove our main homomesy results.

Definition 2.1. For every $i \in [n]$, define $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle map at vertex *i*,** in the following way. If $i \in S$, τ_i removes i from S, which still results in an independent set. If $i \notin S$, then $\tau_i(S)$ adds i to S assuming the resulting set is still independent, and otherwise does nothing. Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

It is clear that each τ_i is an involution, and easy to show the following.

Proposition 2.2. *The toggle maps τ_i and τ_j commute if and only if $|i - j| \neq 1$.*

Proposition 2.3. *When $n \geq 3$, the order of the map $\tau_i \circ \tau_j$ is* $\begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } |i - j| \geq 2 \\ 6 & \text{if } |i - j| = 1 \end{cases}$.

Definition 2.4. Let $\mathfrak{S}_{\mathcal{I}_n}$ denote the symmetric group on $\mathcal{I}_n$. The **toggle group** of $\mathcal{I}_n$, denoted $\mathcal{T}_n$, is the subgroup of $\mathfrak{S}_{\mathcal{I}_n}$ generated by the τ_i toggle maps.

Definition 2.5. A particular element in $\mathcal{T}_n$ is $\varphi := \tau_n \cdots \tau_2 \tau_1$, the map that toggles at each vertex from left to right.

Example 2.6. In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Note that $\varphi^{-1} = \tau_1 \tau_2 \cdots \tau_n$, which applies the toggles from right to left.

Definition 2.7. Given a set $S \in \mathcal{I}_n$ and $j \in [n]$, define $\chi_j(S)$ to be the indicator function of vertex j in S. That is, $\chi_j(S)$ is the j^{th} digit of the binary representation of S. For example, $\chi_1(10010) = 1$, $\chi_2(10010) = 0$, $\chi_3(10010) = 0$, $\chi_4(10010) = 1$, and $\chi_5(10010) = 0$.

The main result is the following, originally conjectured by James Propp.

Theorem 2.8. *Under the action of φ on $\mathcal{I}_n$, $\chi_j - \chi_{n+1-j}$ is 0-mesic for every $1 \leq j \leq n$.*

Definition 2.9. Given an independent set $S \in \mathcal{I}_n$ and $w \in \mathcal{T}_n$, we define the **orbit board** for S and w as follows. Let $S^i = w^i(S)$ and for any $j \in [n]$, let $S(i,j) = 1$ if $j \in S^i$ and $S(i,j) = 0$ if $j \notin S^i$.

Example 2.10. The orbit board for the orbit containing $S = 1010100 \in \mathcal{I}_7$ under the action of φ is shown in Figure 1. This is an orbit of size 15, so $S^{15} = \varphi^{15}(S) = S$. The palindromicity of the columns' sums illustrates Theorem 2.8.

A homomesy result which is much simpler to prove is the following.

Theorem 2.11. *For $n \geq 2$, under the action of φ on $\mathcal{I}_n$, the statistics $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are both 1-mesic.*

The reader can easily check the orbit in Figure 1 for an illustration of Theorem 2.11. This theorem is a corollary of [6, Theorem 7.5] but can also be shown more directly.

	1	2	3	4	5	6	7	8	9	10
S^0	1	0	1	0	1	0	0	1	0	1
S^1	0	0	0	0	0	1	0	0	0	0
S^2	1	0	1	0	0	0	1	0	1	0
S^3	0	0	0	1	0	0	0	0	0	1
S^4	1	0	0	0	1	0	1	0	0	0
S^5	0	1	0	0	0	0	0	1	0	1
S^6	0	0	1	0	1	0	0	0	0	0
S^7	1	0	0	0	0	1	0	1	0	1
S^8	0	1	0	1	0	0	0	0	0	0
S^9	0	0	0	0	1	0	1	0	1	0
S^{10}	1	0	1	0	0	0	0	0	0	1
S^{11}	0	0	0	1	0	1	0	1	0	0
S^{12}	1	0	0	0	0	0	0	0	1	0
S^{13}	0	1	0	1	0	1	0	0	0	1
S^{14}	0	0	0	0	0	0	1	0	0	0
Total	6	3	4	4	4	4	4	4	3	6

Red snake: **221121**

Purple snake: **211212**

Orange snake: **112122**

Green snake: **121221**

Blue snake: **212211**

Brown snake: **122112**

Figure 1: The φ-orbit on $\mathcal{I}_{10}$ that starts with $S = 1010100101$ (See Example 2.17).

We embark on our proof of Theorem 2.8 via a careful analysis of orbit-board properties. For the remainder of this paper, when we refer to $\mathcal{I}_n$, we assume $n \geq 2$.

Lemma 2.12. *1. When $S(i,j) = 1$ and $j \neq n$, either $S(i,j+2) = 1$ or $S(i+1,j+1) = 1$, and never both.*

2. When $S(i,j) = 1$ and $j \neq 1$, either $S(i,j-2) = 1$ or $S(i-1,j-1) = 1$, and never both.

3. *If $S(i,j) = 1$, then $S(i,j-1) = S(i,j+1) = S(i-1,j) = S(i+1,j) = 0$.*

From Lemma 2.12(1), given a 1 in the orbit board (outside of the rightmost column), there is another 1 either in the position two spaces to the right, or the position one space diagonally right and down. From Lemma 2.12(2), for any 1 in the orbit board (outside of the leftmost column), there is another 1 either in the position two spaces to the left, or the position one space diagonally left and up. Therefore, the 1s in the orbit board can be partitioned into sequences, called **snakes**, that begin in the left column and end in the right column. For any 1 in the snake, the next 1 is located either two spaces to the right of it, or in the position one space diagonally right and down.

Therefore, to know where the 1s in the orbit board are, it suffices to analyze the snakes. To each φ-orbit on $\mathcal{I}_n$, we will associate an equivalence class of compositions of $n-1$ into parts 1 and 2, with each composition representing the snakes.

Definition 2.13. A **composition** of $n \in \mathbb{Z}^+$ is a sequence of positive integers that add to n. Two compositions of n are said to be **cyclically equivalent** if one is a cyclic rotation of the other. Otherwise, the compositions are **cyclically inequivalent**.

Example 2.14. 21121, 11212, 12121, 21211, and 12112 are cyclically equivalent compositions of 7, and all are cyclically inequivalent to 22111.

To associate a composition of $n-1$ to any given snake in a φ-orbit of $\mathcal{I}_n$, a step of two positions to the right corresponds to a 2, and a step of one position diagonally right and down corresponds to a 1. Therefore, 1 and 2 represent the number of positions to the right. Thus, the reason we get a composition of $n-1$ is because we start in the leftmost column and end in the rightmost column.

Definition 2.15. The **snake composition** for a snake is the composition that corresponds to the snake in the way just described.

Refer to Figure 1 for the compositions that correspond to the six snakes. Analyzing where 1s can be positioned on orbit boards leads to the following theorem.

Theorem 2.16. *In an orbit, consider a snake starting on the S^i line. Let c be the snake's composition. Consider the least $i' > i$ for which $S(i',1) = 1$. (This is where the "next" snake begins.)*

1. *If c starts with 1, then $i' = i+3$.*

2. *If c starts with 2, then $i' = i+2$.*

3. *The composition for the snake starting on the $S^{i'}$ line is a left cyclic rotation of c.*

Example 2.17. We show how knowing one snake determines an entire orbit. Suppose we are working in $\mathcal{I}_{10}$ and we have a snake given by the composition 221121. This gives us the red snake in Figure 1. We can construct the entire orbit from this snake.

Using Theorem 2.16, we know that the next snake begins on the S^2 line, and has snake composition 211212. This snake is shown in purple in the figure.

Also by Theorem 2.16, the next four snakes start on the lines have snake compositions 112122, 121221, 212211, and 122112, and begin on lines S^4, S^7, S^{10} and S^{12}. These are shown in orange, green, blue, and brown respectively in Figure 1.

Then the next snake starts on the S^{15} line and has snake composition 221121. However, this is the snake we started with. Therefore, $S^0 = S^{15}$, this orbit has size 15, and the 1s in the brown snake in the board above go on the S^0 line. Every other empty position is a 0 by Lemma 2.12(3). This gives us the entire orbit.

The following should be clear now.

Proposition 2.18. *The snake compositions of the snakes in any orbit are cyclic rotations of each other. Thus, there is a bijection between φ-orbits of $\mathcal{I}_n$ and cyclically inequivalent compositions of $n-1$ into parts 1 and 2.*

The proof of Theorem 2.8 follows from this snake description. Since every snake in $\mathcal{O}$ starts in the leftmost column and ends in the rightmost column, the orbit has the same number of 1s in the leftmost column as in the rightmost column of the (finite version of the) orbit board. There is a 1 in column j exactly when a snake's composition has an initial segment that adds to $j-1$, and similarly a 1 in column $n+1-j$ exactly when a snake's composition has a final segment that adds to $j-1$. By cyclic rotation, for any orbit, there are the same number of snakes with an initial segment that adds to $j-1$ as there are with a final segment that adds to $j-1$.

Next we indicate how to generalize our results to other products of toggles. As $\mathcal{T}_n$ is generated by finitely many involutions, it is the quotient of a Coxeter group; see [2] and [10, Ch. 11-14]. Even though $\mathcal{T}_n$ is not a true Coxeter group, we borrow a term from Coxeter group theory to describe an important class of elements in $\mathcal{T}_n$.

Definition 2.19. An element $w \in \mathcal{T}_n$ is called a **Coxeter element** if it is a product of $\tau_1, \tau_2, \ldots, \tau_n$, each used exactly once, in some order.

Any two Coxeter elements in $\mathcal{T}_n$ are conjugate [19, Lemma 5.1]. We can use the conjugation to prove the following. (More details in the forthcoming paper.)

Theorem 2.20. *Let $w, w' : \mathcal{I}_n \to \mathcal{I}_n$ be two Coxeter elements in $\mathcal{T}_n$. Then any statistic which is a linear combination of the indicator functions χ_j is c-mesic under the action of w if and only if it is c-mesic under the action of w'.*

The following theorem generalizes Theorems 2.8 and 2.11 to Coxeter elements in $\mathcal{T}_n$.

Theorem 2.21. *Let $w \in \mathcal{T}_n$ be a Coxeter element. Under the action of w on $\mathcal{I}_n$, $\chi_j - \chi_{n+1-j}$ is 0-mesic for every $1 \le j \le n$. Also, $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ are both 1-mesic.*

3 Counting Orbits

In this section, we discuss enumeration of orbits. It is well known and easy to prove that independent sets of $\mathcal{P}_n$ are counted by Fibonacci numbers. Specifically, $\#\mathcal{I}_n = F_{n+2}$ where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. The authors used OEIS [14] before proving Theorem 2.8 and discovered the number of φ-orbits of $\mathcal{I}_n$ is the sequence A000358 (with the index shifted by 1), which interestingly is also expressed in terms of Fibonacci numbers. This eventually led to the description of snakes and the proof of the main theorem. Specifically, the number of cyclically inequivalent compositions of n with each part equal to 1 or 2 is $\frac{1}{n}\sum\limits_{d|n} \varphi(n/d)(F_{d-1} + F_{d+1})$ [14, A000358]. Therefore, we get a formula for the number of φ-orbits on $\mathcal{I}_n$.

Theorem 3.1. *The total number of φ-orbits of $\mathcal{I}_n$ is* $\frac{1}{n-1}\sum\limits_{d|(n-1)} \varphi((n-1)/d)(F_{d-1} + F_{d+1})$.

We also get the size of a φ-orbit from the corresponding cyclic equivalence class of snake compositions. Whenever the snake composition contains a 2, the next snake starts two positions down, and when a snake composition contains a 1, the next snake starts three positions down all by Theorem 2.16. The intuitive idea is that given a snake composition in the orbit, such as 221121, change each 1 to a 3 and sum the parts. Thus, the orbit shown in Example 2.17 has size $2 + 2 + 3 + 3 + 2 + 3 = 15$, which can be seen in that example. This approach can fail when we generate a superorbit instead of an orbit. For example, consider the orbit of $\mathcal{I}_7$ given by snake composition 2121. This orbit has size 5, not $2 + 3 + 2 + 3 = 10$, because of the symmetry in 2121, which only has two distinct compositions in its cyclic equivalence class. Therefore, given a snake composition such as 2121 made up entirely of a repeated segment (in this case 21), we must divide by the number of times the minimal repeated segment repeats itself in the string (in this case 2). So the orbit size is $\frac{2+3+2+3}{2} = 5$.

Definition 3.2. Call a composition c **periodic** if it consists of adjacent copies of the same repeated substring. let $\psi(c)$ denote the number of times the smallest repeated segment repeats itself to make up c.

Example 3.3. For the composition $c_1 = 21221$, $\psi(c_1) = 1$. For the composition $c_2 = 22122212$, $\psi(c_2) = 2$. For the composition $c_3 = 222$, $\psi(c_3) = 3$.

Theorem 3.4. *Given a φ-orbit $\mathcal{O}$ containing snake composition c, let $N_1(c)$ be the number of occurrences of 1 in c and $N_2(c)$ be the number of occurrences of 2 in c. Then the size of the orbit $\mathcal{O}$ is* $\frac{3N_1(c)+2N_2(c)}{\psi(c)}$.

Therefore, given any orbit size, we can characterize exactly for which $n \geq 2$, there is an orbit of $\mathcal{I}_n$ with that size, and how many such orbits.

Example 3.5. The only composition of 2 into parts 2 and 3 is the composition 2. Therefore, an orbit has size 2 if and only if a snake composition corresponding to the orbit is of the form $222\cdots2$, with k 2s repeated. This snake composition is in an orbit of $\mathcal{I}_{2k+1}$. So there is an orbit of size 2 if and only if n is odd, and this orbit is unique. It can be easily shown that this orbit consists of $\emptyset$ and $\{1,3,5,\ldots,n\}$.

The only composition of 3 into parts 2 and 3 is the composition 3. Therefore, a φ-orbit has size 3 if and only if a snake composition corresponding to the orbit is of the form $111\cdots1$, with k 1s repeated. This snake composition is in an orbit of $\mathcal{I}_{k+1}$. Thus, there exists a unique orbit of size 3 for all $n \geq 2$.

The only composition of 4 into parts 2 and 3 is the composition $2+2$. However, this composition is made up entirely of a smaller repeated pattern, and therefore gives an orbit of size 2. Thus, there are no orbits of size 4. An analogous argument shows there are no orbits of size 6.

The only compositions of 5 into parts 2 and 3 are $2+3$ and $3+2$, which are cyclically equivalent. Therefore, an orbit has size 5 if and only if a snake composition corresponding to the orbit is of the form $1212\cdots12$, with k total patterns of 12 repeated, so this is a composition of $3k$. So there is an orbit of size 5 if and only if $n \equiv 1 \bmod 3$.

The following proposition holds because two iterations of φ show that the orbit contains a snake with composition $222\cdots21$, with $\frac{n-1}{2}$ total 2s.

Proposition 3.6. *For even n, the φ-orbit of $\mathcal{I}_n$ containing the empty set has size $n+1$.*

Theorem 3.7. *Let $\mathcal{O}$ be an φ-orbit of $\mathcal{I}_n$ and c be a snake composition that appears in $\mathcal{O}$. If $\psi(c) = 1$, then the size of $\mathcal{O}$ is congruent to $1-n \bmod 4$. Furthermore, regardless of $\psi(c)$, the size of $\mathcal{O}$ divides an integer $m \equiv 1-n \bmod 4$ for $m \leq 3(n-1)$ (where m depends on $\mathcal{O}$).*

Corollary 3.8. *For even n, every φ-orbit of $\mathcal{I}_n$ has odd size. Furthermore, when $n \equiv 3 \bmod 4$, there also exist no orbits with size divisible by 4.*

4 Connections with Order Ideals in Zigzag Posets

The original problem about independent sets is connected with other well-studied maps, called *promotion* and *rowmotion*, on zigzag posets. Rowmotion was introduced as a map on antichains in [3], but is equivalently a map on order ideals. Promotion and rowmotion have both been studied in various settings by many authors, as well summarized in [19].

We assume the reader is familiar with basic poset theory. Refer to [15, Ch. 3] for a detailed introduction to posets. We are interested in a special class of posets, called *zigzag posets* whose name comes from the shape of their Hasse diagrams.

Definition 4.1. The **zigzag poset** with n elements, denoted $\mathcal{Z}_n$, is the poset consisting of elements $a_1, ..., a_n$ and relations $a_{2i-1} < a_{2i}$ and $a_{2i+1} < a_{2i}$ [15, p. 367].

Definition 4.2. An **order ideal** of a poset P is a subset I of P such that if $x \in I$ and $y < x$ in P, then $y \in I$. The set of order ideals of P is denoted $J(P)$.

The toggle group on $J(\mathcal{Z}_n)$ is defined analogously to that of $\mathcal{I}_n$.

Definition 4.3. The **toggle map** $\tau_i : J(\mathcal{Z}_n) \to J(\mathcal{Z}_n)$ is defined as

$$t_i(I) := \begin{cases} I\Delta\{a_i\} & \text{if } I\Delta\{a_i\} \in J(\mathcal{Z}_n) \\ I & \text{if } I\Delta\{a_i\} \notin J(\mathcal{Z}_n) \end{cases}$$

where $I\Delta\{a_i\} := (I \setminus \{a_i\}) \cup (\{a_i\} \setminus I)$ is the symmetric difference of I and $\{a_i\}$. The **toggle group** of $J(\mathcal{Z}_n)$, denoted $\mathrm{Tog}(\mathcal{Z}_n)$, is the group generated by the toggle maps t_i for $i \in [n]$.

As with toggle maps on $\mathcal{I}_n$, $t_i^2 = 1$ for any $i \in [n]$ and $t_i, t_j \in \mathrm{Tog}(\mathcal{Z}_n)$ commute if and only if $|i - j| \neq 1$.

Two special elements of $\mathrm{Tog}(\mathcal{Z}_n)$ are **promotion** $\mathrm{Pro} = t_n \cdots t_2 t_1$ and **rowmotion**

$$\mathrm{Row} = \begin{cases} t_{n-1}t_{n-3}\cdots t_3t_1t_nt_{n-2}\cdots t_2 & \text{if } n \text{ is even} \\ t_nt_{n-2}\cdots t_3t_1t_{n-1}t_{n-3}\cdots t_2 & \text{if } n \text{ is odd} \end{cases}.$$

These maps have been studied on general posets by numerous authors. Rowmotion can be defined for any poset and promotion can be defined for any rowed-and-columned poset (which $\mathcal{Z}_n$ is) [19].

Independent sets of $\mathcal{P}_n$ are in bijection with order ideals of $\mathcal{Z}_n$.

Proposition 4.4. *There is a bijection $\eta : \mathcal{I}_n \to J(\mathcal{Z}_n)$ defined as*

$$\eta(S) := \{a_i \mid i \in [n], i \text{ odd}, i \notin S\} \cup \{a_i \mid i \in [n], i \text{ even}, i \in S\}.$$

Example 4.5. Let $n = 7$ and $S = 1001010 = \{1, 4, 6\}$. Then $a_1 \notin \eta(S)$ and $a_3, a_5, a_7 \in \eta(S)$ because $1 \in S$ and $3, 5, 7 \notin S$. Also, $a_2 \notin \eta(S)$ and $a_4, a_6 \in \eta(S)$, since $2 \notin S$ and $4, 6 \in S$. So $\eta(S) = \{a_3, a_4, a_5, a_6, a_7\}$.

This map η is an equivariant bijection with respect to the toggle maps. This is described in the proposition below, which is obvious from the way the maps are defined.

Proposition 4.6. *For every $i \in [n]$, $\eta \circ \tau_i = t_i \circ \eta$. Thus, $\eta \circ \varphi = \mathrm{Pro} \circ \eta$. See the commutative diagrams below.*

$$\begin{tikzcd} \mathcal{I}_n \arrow[r, "\eta"] \arrow[d, "\tau_i"'] & J(\mathcal{Z}_n) \arrow[d, "t_i"] \\ \mathcal{I}_n \arrow[r, "\eta"'] & J(\mathcal{Z}_n) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[r, "\eta"] \arrow[d, "\varphi"'] & J(\mathcal{Z}_n) \arrow[d, "\mathrm{Pro}"] \\ \mathcal{I}_n \arrow[r, "\eta"'] & J(\mathcal{Z}_n) \end{tikzcd}$$

As with $\mathcal{T}_n$, $\text{Tog}(\mathcal{Z}_n)$ is the quotient of a Coxeter group, and it is clearly isomorphic to $\mathcal{T}_n$ via Proposition 4.6. We define Coxeter elements analogously to $\mathcal{T}_n$, and Pro and Row are two examples of Coxeter elements in $\text{Tog}(\mathcal{Z}_n)$.

Definition 4.7. A **Coxeter element** in $\text{Tog}(\mathcal{Z}_n)$ is a product of each of the n toggle maps $t_1, t_2, \ldots, t_n$ each used exactly once, in some order.

Theorem 4.8. *Any two Coxeter elements in* $\text{Tog}(\mathcal{Z}_n)$ *are conjugate.*

Via Proposition 4.6, we can restate Theorem 2.21 for toggling in $J(\mathcal{Z}_n)$, as follows.

Theorem 4.9. *Let w be a Coxeter element in* $\text{Tog}(\mathcal{Z}_n)$. *Let $\chi_{a_j} : J(Z_n) \to \{0,1\}$ be the indicator function of a_j. Then on w-orbits in $J(Z_n)$, the following statistics are homomesic.*

- *If n is odd, then $\chi_{a_j} - \chi_{a_{n+1-j}}$ is 0-mesic for every $j \in [n]$. Also $2\chi_{a_1} - \chi_{a_2}$ and $2\chi_{a_n} - \chi_{a_{n-1}}$ are both 1-mesic.*
- *If n is even, then $\chi_{a_j} + \chi_{a_{n+1-j}}$ is 1-mesic for every $j \in [n]$. Also $2\chi_{a_1} - \chi_{a_2}$ is 1-mesic and $2\chi_{a_n} - \chi_{a_{n-1}}$ is 0-mesic.*

Notice that our statements above are significantly more complicated to state, forcing us to divide into odd and even cases. This would also make direct proofs of them in the $J(\mathcal{Z}_n)$-setting more unwieldy. It is much easier to handle them via translation to the $\mathcal{I}_n$ context. This shows the efficacy of Striker's notion of generalized toggling.

It is well-known and not hard to see that for any graded poset P of rank r, there is a rowmotion orbit on $J(P)$ of size $r+1$ generated by the empty ideal, where $\text{Row}^i(\varnothing)$ consists of all elements of rank $\leq i-1$. In particular, $J(\mathcal{Z}_n)$ has a rowmotion orbit of size 3. It is not directly obvious that this is the only orbit of this size. But since the orbit structure of Row is the same as that of φ on $\mathcal{I}_n$, uniqueness follows from the discussion in Section 3 and the equivariant bijection of Proposition 4.6.

In other proven examples of homomesy for rowmotion on posets, the map generally has a small order and the cyclic sieving phenomenon has been found. However, the rowmotion map on $J(\mathcal{Z}_n)$ has a large order, and thus a natural cyclic sieving result is unlikely, which makes the homomesy for this poset interesting.

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